20. Convolution: Graphical Solution

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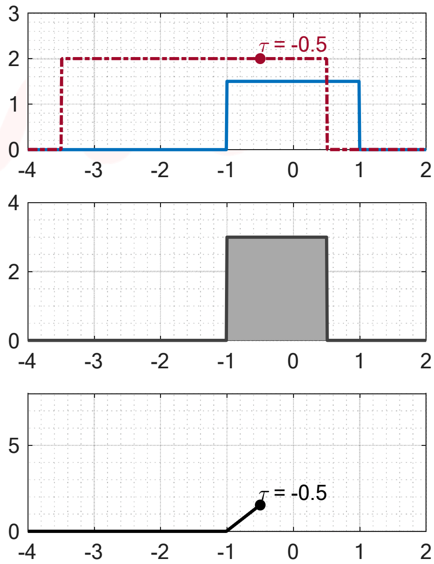


Graphical convolution procedure:

$$z(\tau) = x(t) * y(t) = \int_{-\infty}^{\infty} x(t)\, y(\tau - t)\, dt$$

1. Draw the function $x(t)$ as is (without any modifications).

2. Draw $y(t)$, then draw its time-inverted version $y(-t)$, which is simply a rotation (flip) around the vertical axis.

3. To calculate $z(\tau)$ at instant τ, time-shift the new signal $y(-t)$ by an amount of time $-\tau$ to get $y\big(-(t-\tau)\big)$ (e.g., to find $z(\tau = 4)$, draw $y\big(-(t-4)\big)$, i.e., shift $y(-t)$ to the right by 4 seconds). Remember to shift $y(-t)$, NOT the original $y(t)$. This gives $y\big(-(t-\tau)\big) = y(\tau - t)$.

$$z(\tau) = x(t) * y(t) = \int_{-\infty}^{\infty} x(t)\, y(\tau - t)\, dt$$

4. Multiply the signals you got from step 1 (which is $x(t)$) and step 3 (which is $y(\tau - t)$) to get the new product signal $g(t) = x(t)\, y(\tau - t)$.

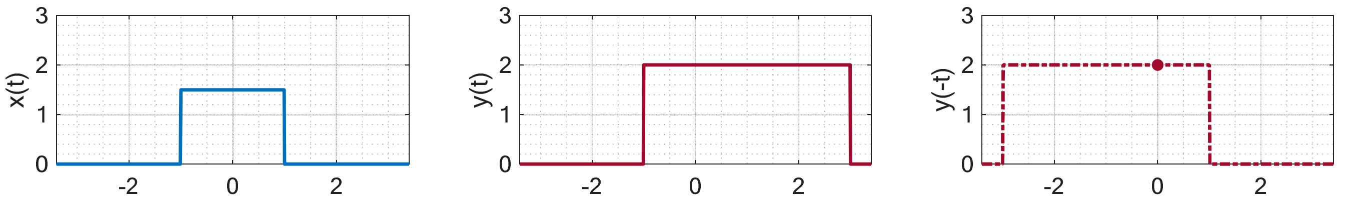
5. Calculate the area under the product $g(t) = x(t)\, y(\tau - t)$ (i.e., the integral). This area is the value of $z(\tau) = (x * y)(\tau)$ at time shift τ.

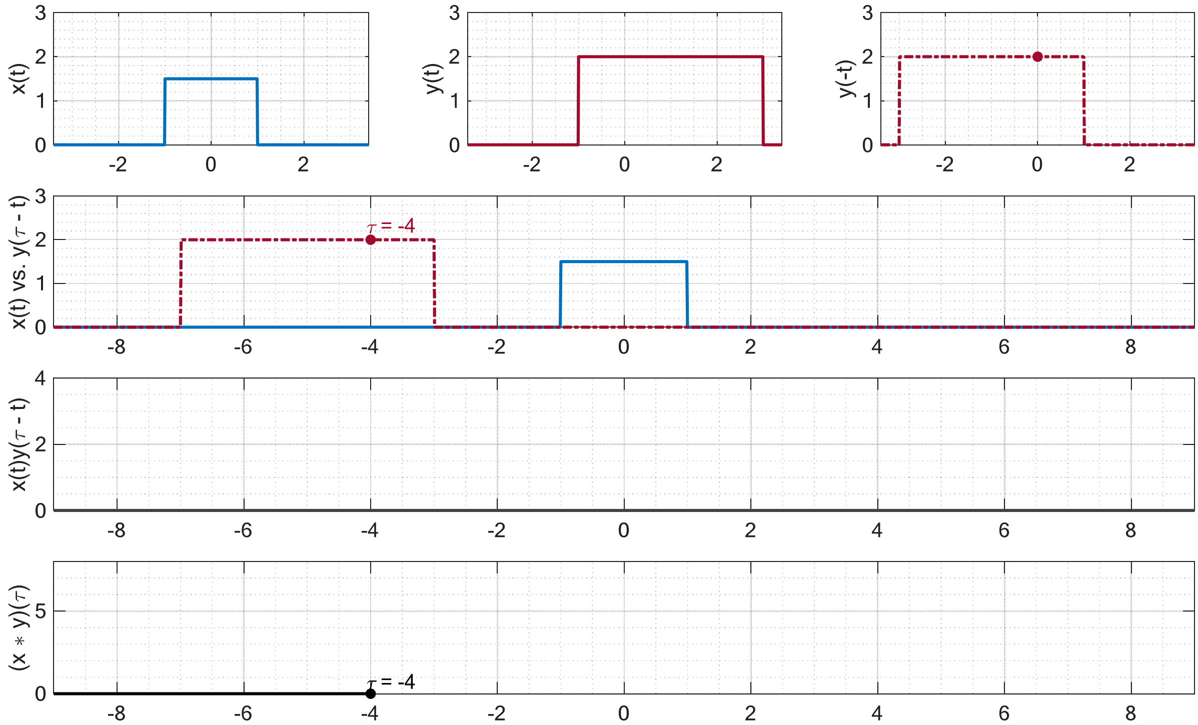
6. Repeat the above procedure, shifting $y(-t)$ by different time shifts $\tau \in (-\infty, \infty)$ to get $z(\tau)$ for all $\tau \in (-\infty, \infty)$: One shift for each value.

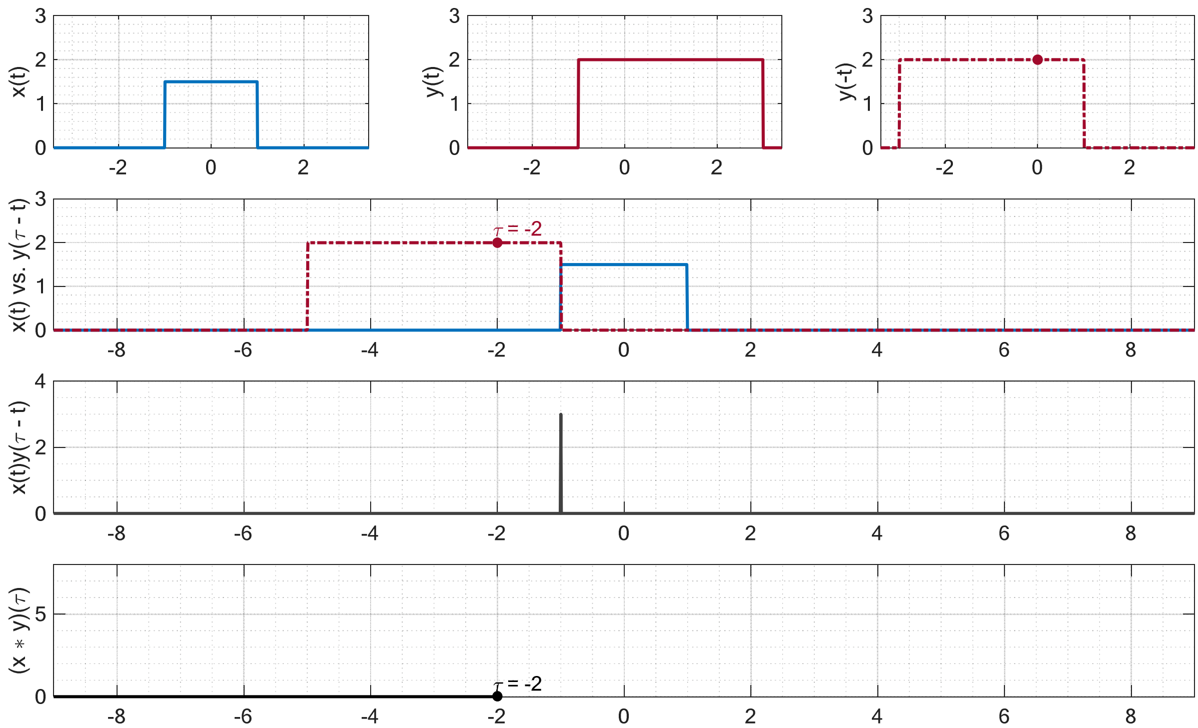
7. Instead of calculating $z(\tau)$ for infinity points $\tau \in (-\infty, \infty)$, we typically break τ into ranges (e.g., $\tau \in (-\infty, -2]$, $\tau \in (-2, 0]$, $\tau \in (0, 1]$ and $\tau \in (1, \infty)$), and calculate $z(\tau)$ as an equation for each region.

Q3. For the signals $x(t) = 1.5 \text{ rect}\left(\frac{t}{2}\right)$ and $y(t) = 2 \text{ rect}\left(\frac{t-1}{4}\right)$, determine the convolution result $x(t) * y(t)$.

Q3. Solution. To perform the graphical solution, first draw $x(t)$, $y(t)$ and then $y(-t)$ as follows




x(t)
y(t)
y(-t)
x(t) vs. y(τ - t)
τ = -4
x(t)y(τ - t)
(x * y)(τ)
τ = -4


x(t)
y(t)
y(-t)
x(t) vs. y(τ - t)
τ = -2
x(t)y(τ - t)
(x * y)(τ)
τ = -2

Region #1: For time-shift $\tau \leq -2$

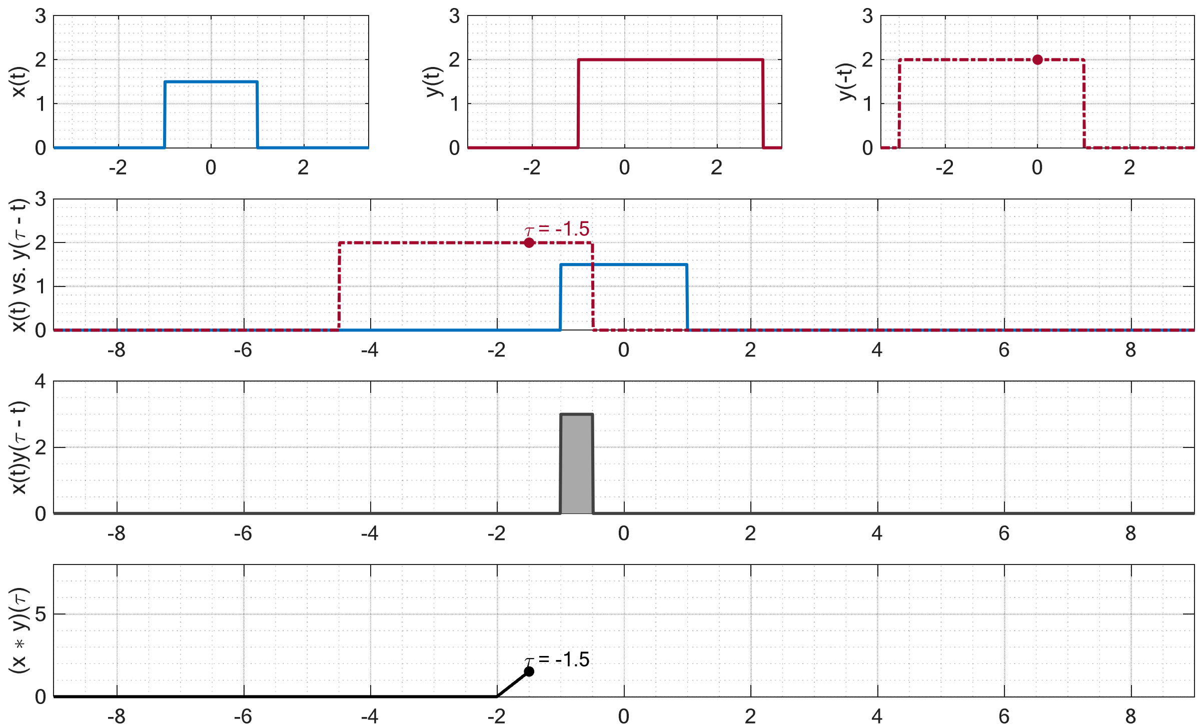
$$z(\tau) = x(t) * y(t) = \int_{-\infty}^{\infty} x(t)\, y(\tau - t)\, dt$$

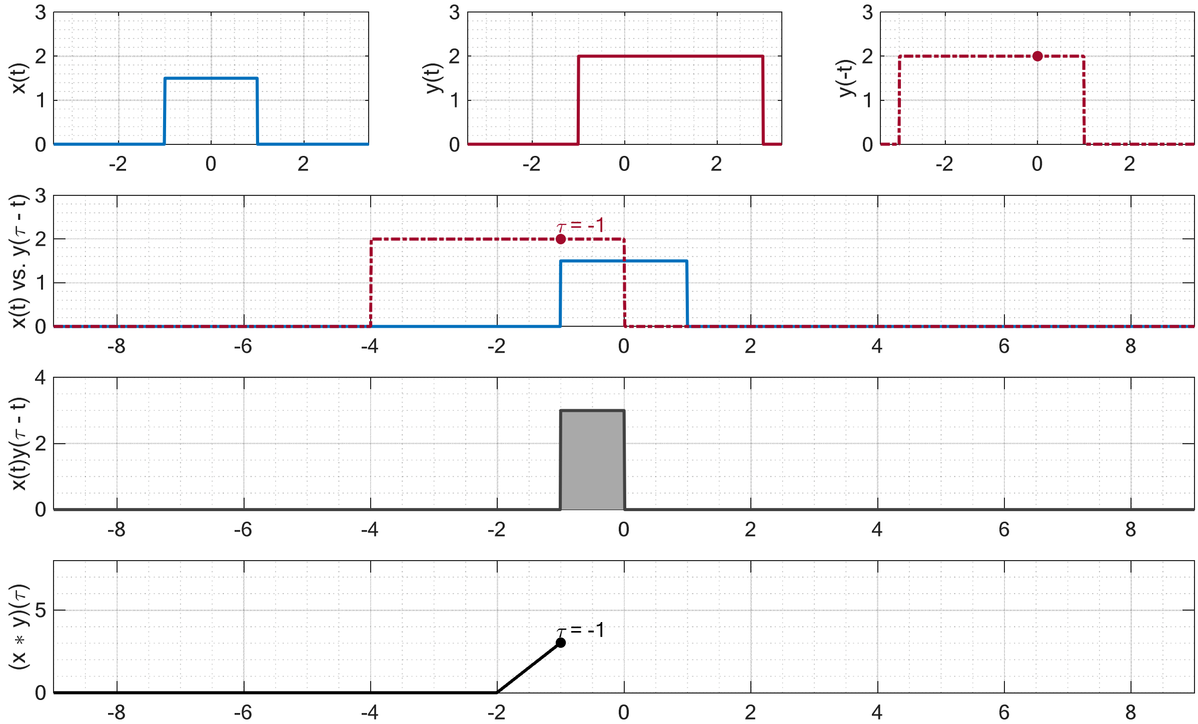
$$z(\tau) = \int_{-\infty}^{\infty} (0)\, dt = 0$$

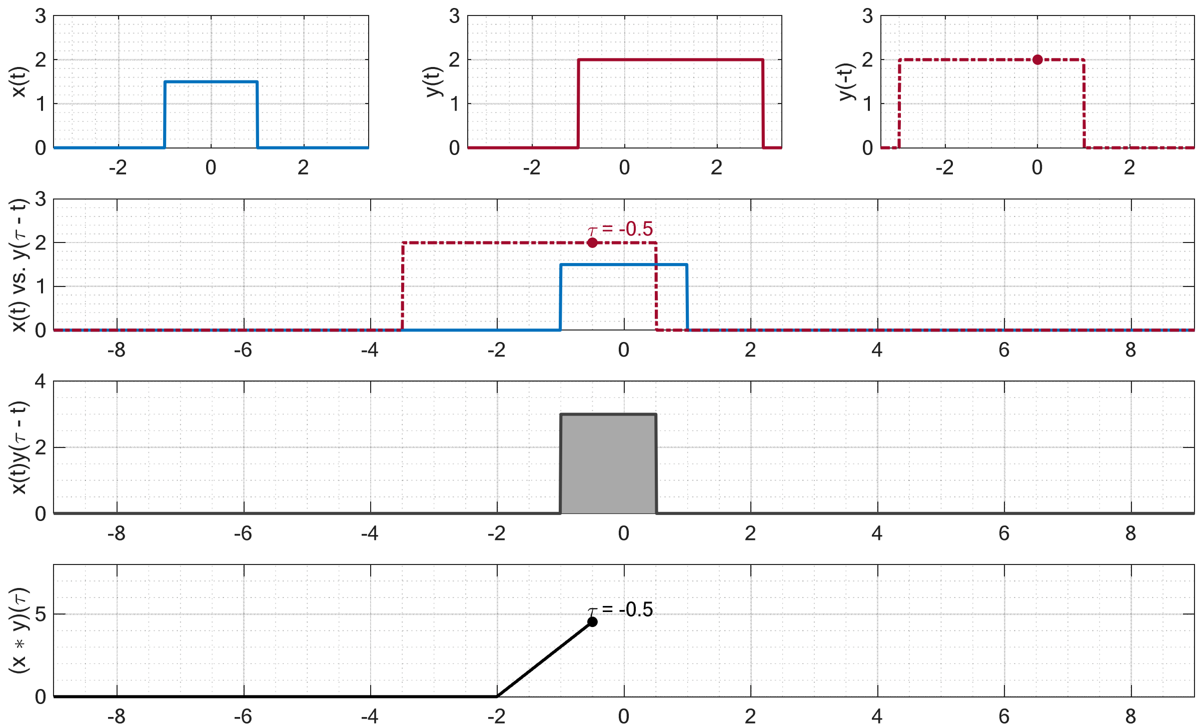
Region #2: For time-shift $-2 < \tau \leq 0$

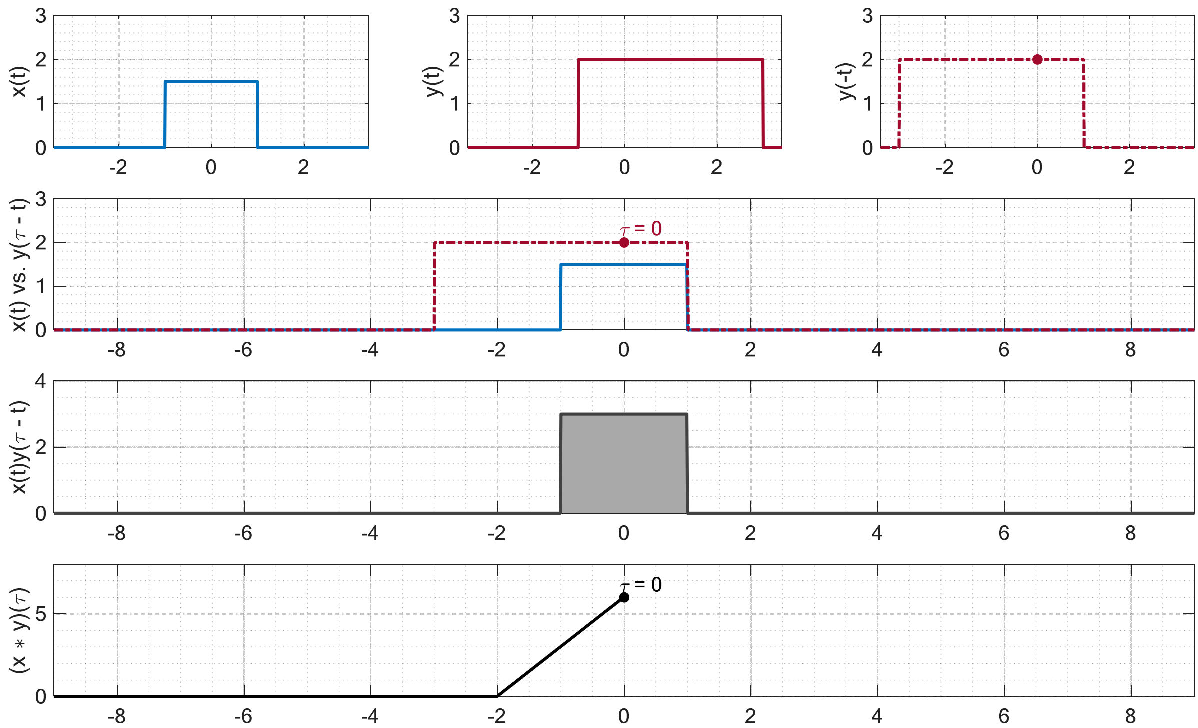
$$z(\tau) = x(t) * y(t) = \int_{-\infty}^{\infty} x(t)\, y(\tau - t)\, dt$$

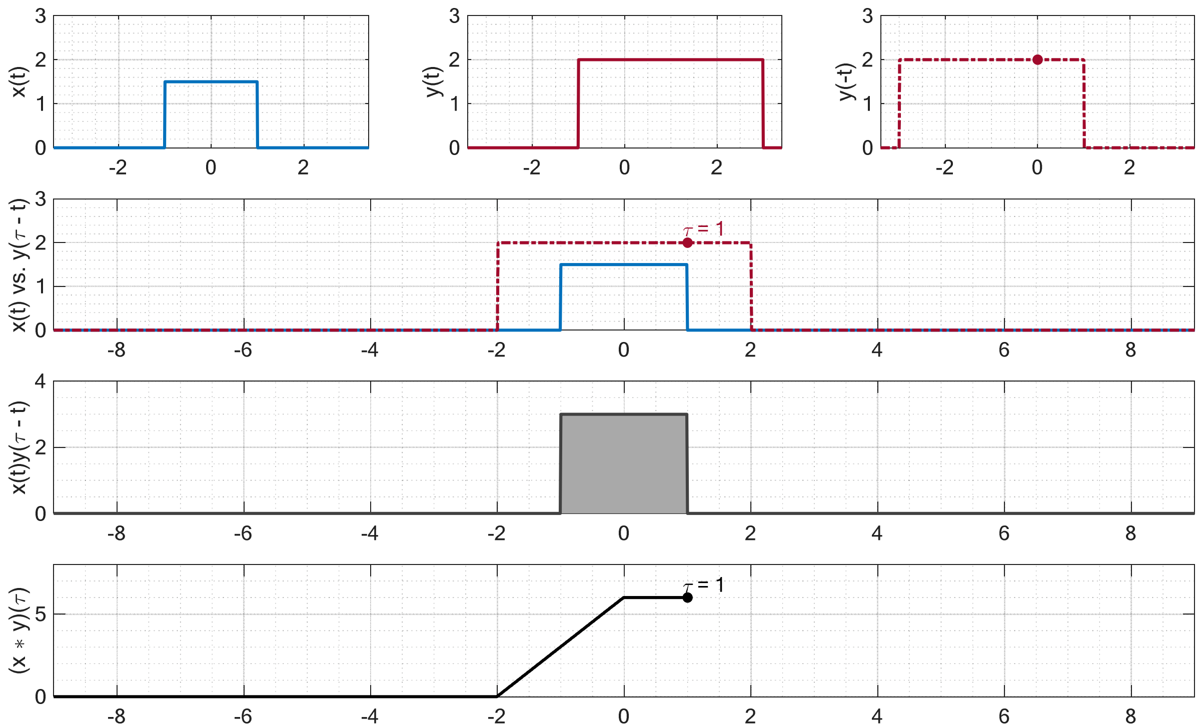
$$z(\tau) = \int_{-1}^{\tau+1} 1.5 \times 2\, dt = 1.5 \times 2 \times [t]_{-1}^{\tau+1} = 3(\tau + 2) = 3\tau + 6$$

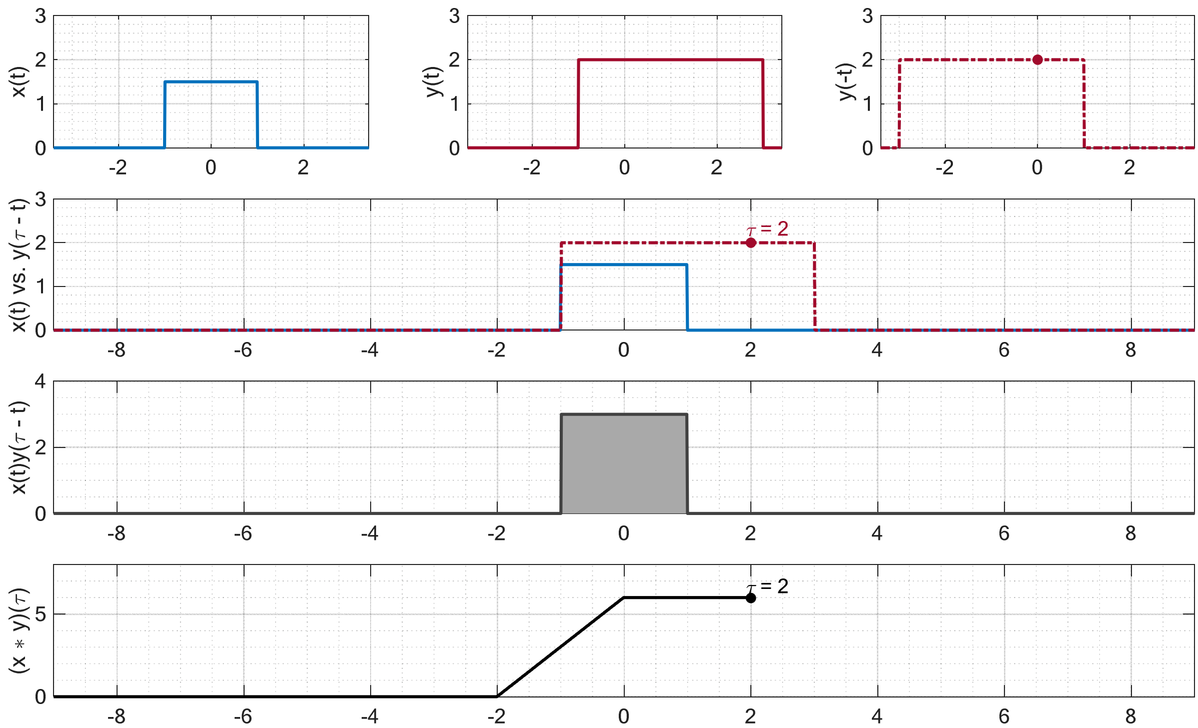



x(t)
y(t)
y(-t)
x(t) vs. y(τ - t)
τ = -1
x(t)y(τ - t)
(x * y)(τ)
τ = -1


x(t)
y(t)
y(-t)
x(t) vs. y(τ - t)
τ = -0.5
x(t)y(τ - t)
(x * y)(τ)
τ = -0.5


x(t)
y(t)
y(-t)
x(t) vs. y(τ - t)
τ = 0
x(t)y(τ - t)
(x * y)(τ)
τ = 0


x(t)
y(t)
y(-t)
x(t) vs. y(τ - t)
τ = 1
x(t)y(τ - t)
(x * y)(τ)
τ = 1



Region #3: For time-shift $0 < \tau \leq 2$

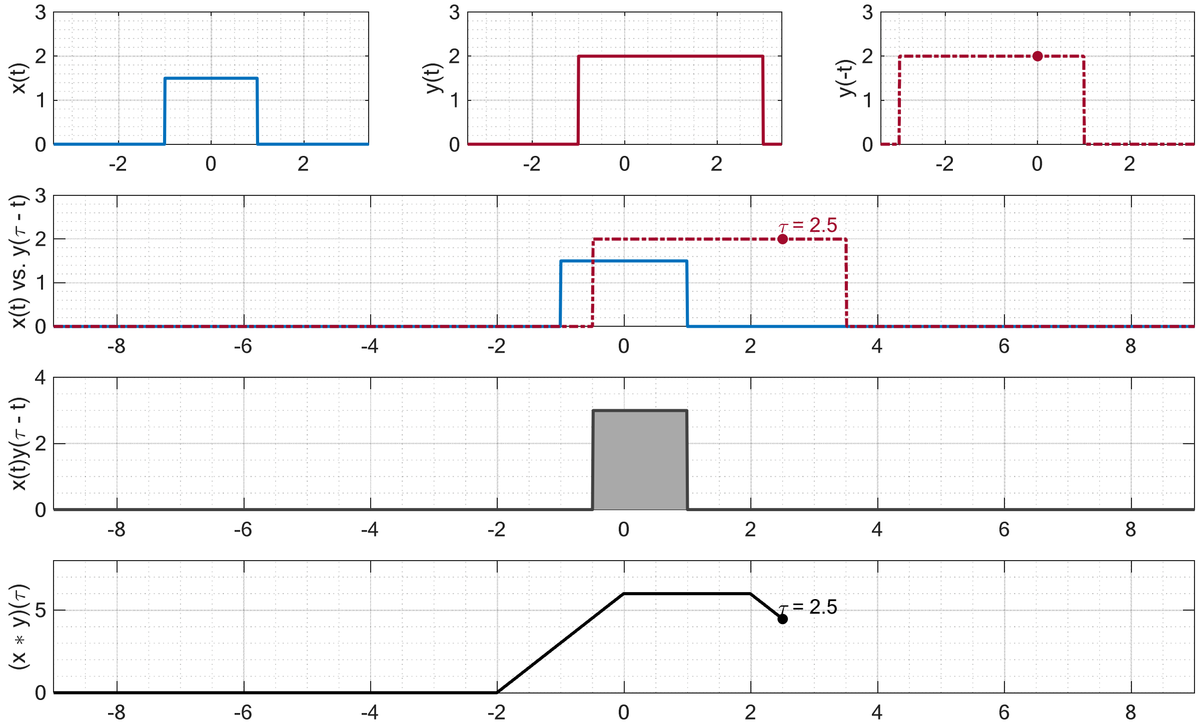
$$z(\tau) = x(t) * y(t) = \int_{-\infty}^{\infty} x(t)\, y(\tau - t)\, dt$$

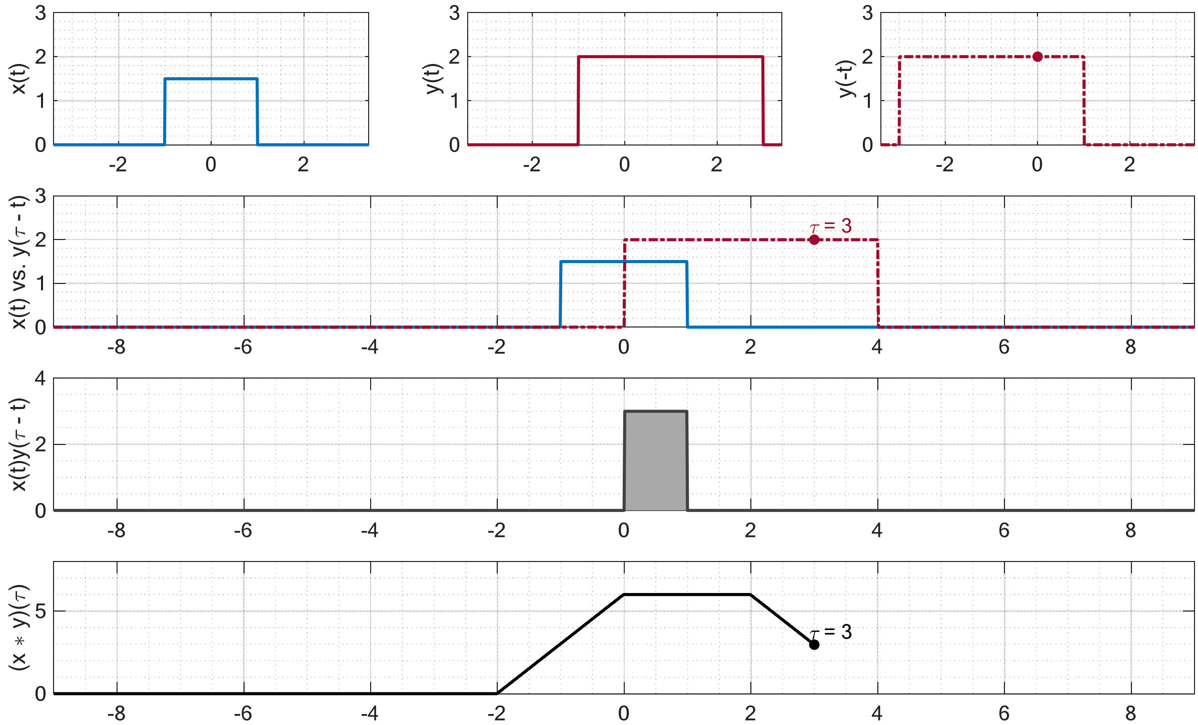
$$z(\tau) = \int_{-1}^{1} (1.5 \times 2)\, dt = 1.5 \times 2 \times [t]_{-1}^{1} = 3 \times (2) = 6$$

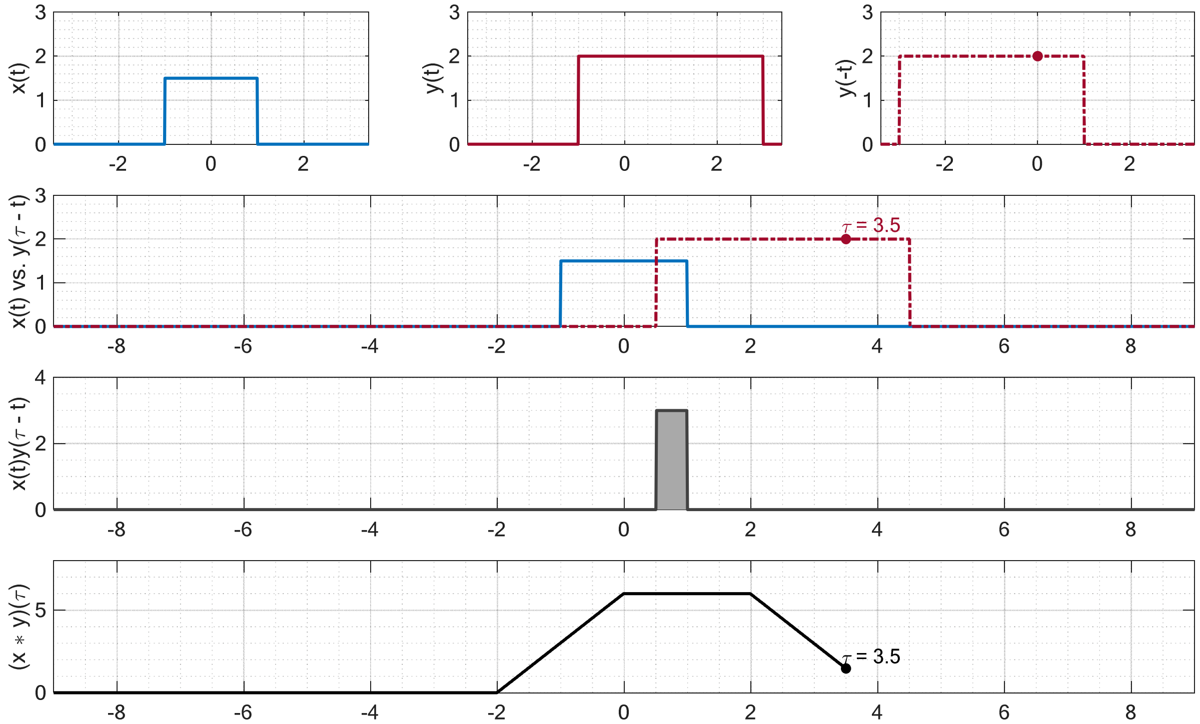
Region #4: For time-shift $2 < \tau \leq 4$

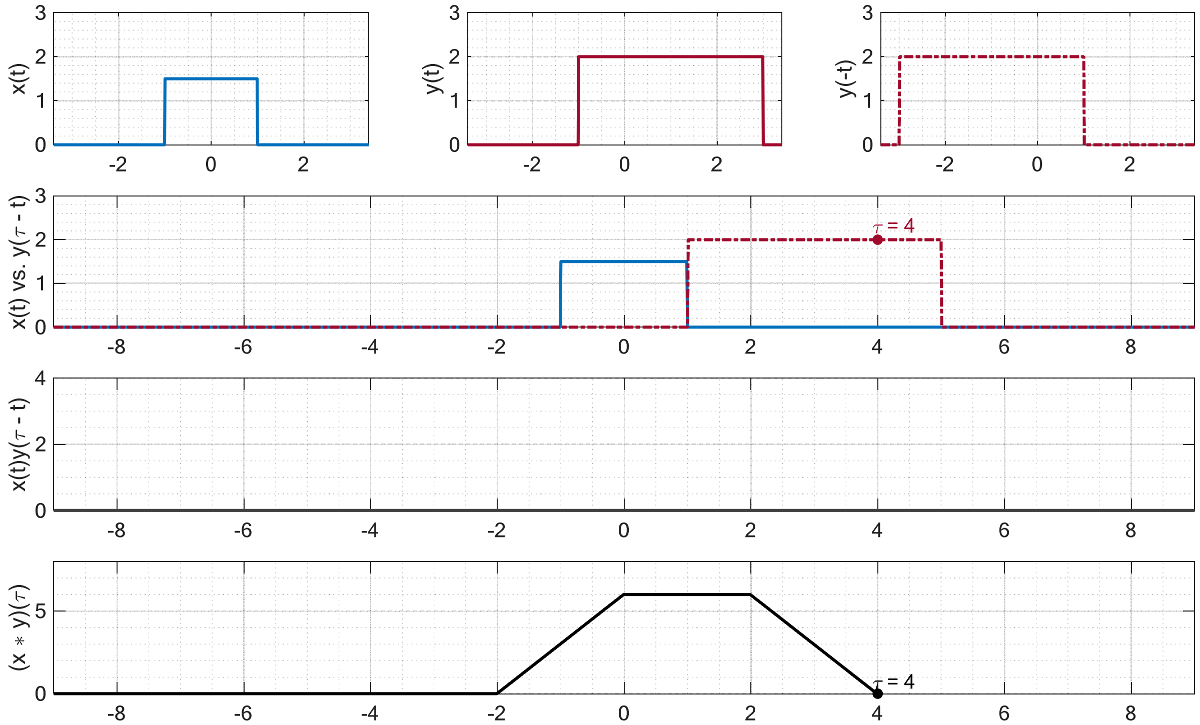
$$z(\tau) = x(t) * y(t) = \int_{-\infty}^{\infty} x(t)\, y(\tau - t)\, dt$$

$$z(\tau) = \int_{\tau-3}^{1} 1.5 \times 2\, dt = 1.5 \times 2 \times [t]_{\tau-3}^{1} = 3(-\tau + 4) = -3\tau + 12$$


x(t)
y(t)
y(-t)
x(t) vs. y(τ - t)
τ = 2.5
x(t)y(τ - t)
(x * y)(τ)
τ = 2.5


x(t)
y(t)
y(-t)
x(t) vs. y(τ - t)
τ = 3
x(t)y(τ - t)
(x * y)(τ)
τ = 3


x(t)
y(t)
y(-t)
x(t) vs. y(τ - t)
τ = 3.5
x(t)y(τ - t)
(x * y)(τ)
τ = 3.5


x(t)
y(t)
y(-t)
x(t) vs. y(τ - t)
τ = 4
x(t)y(τ - t)
(x * y)(τ)
τ = 4

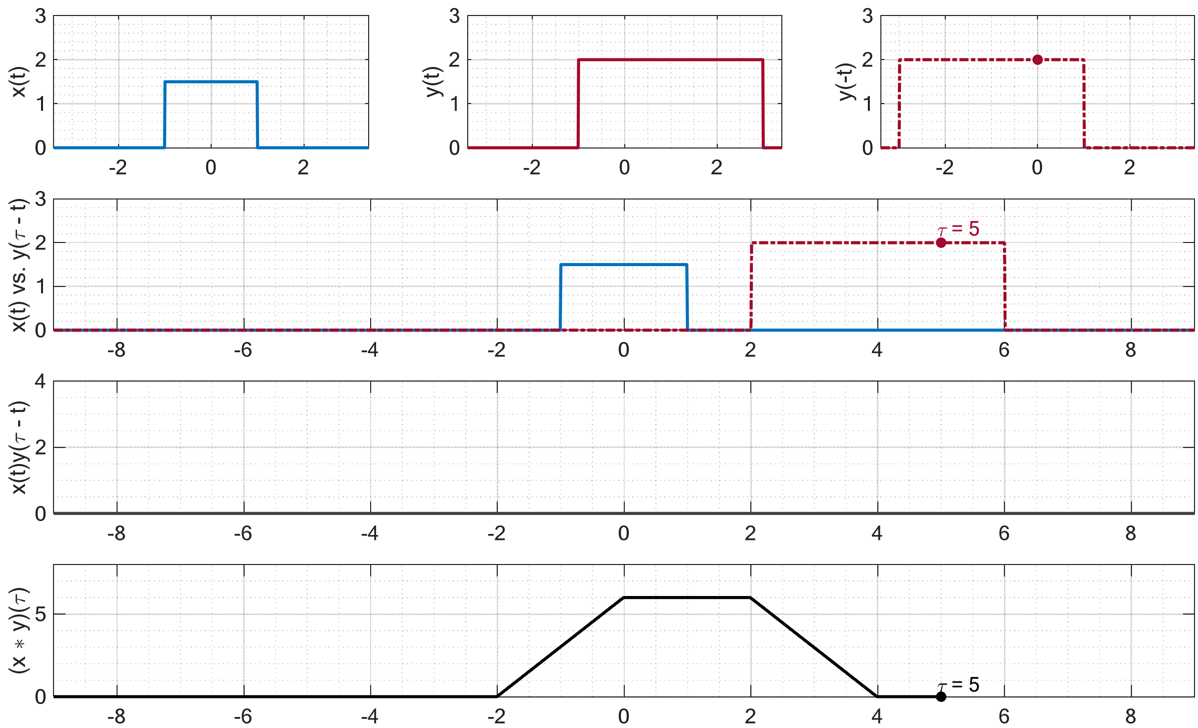
Region #5: For time-shift $\tau > 4$

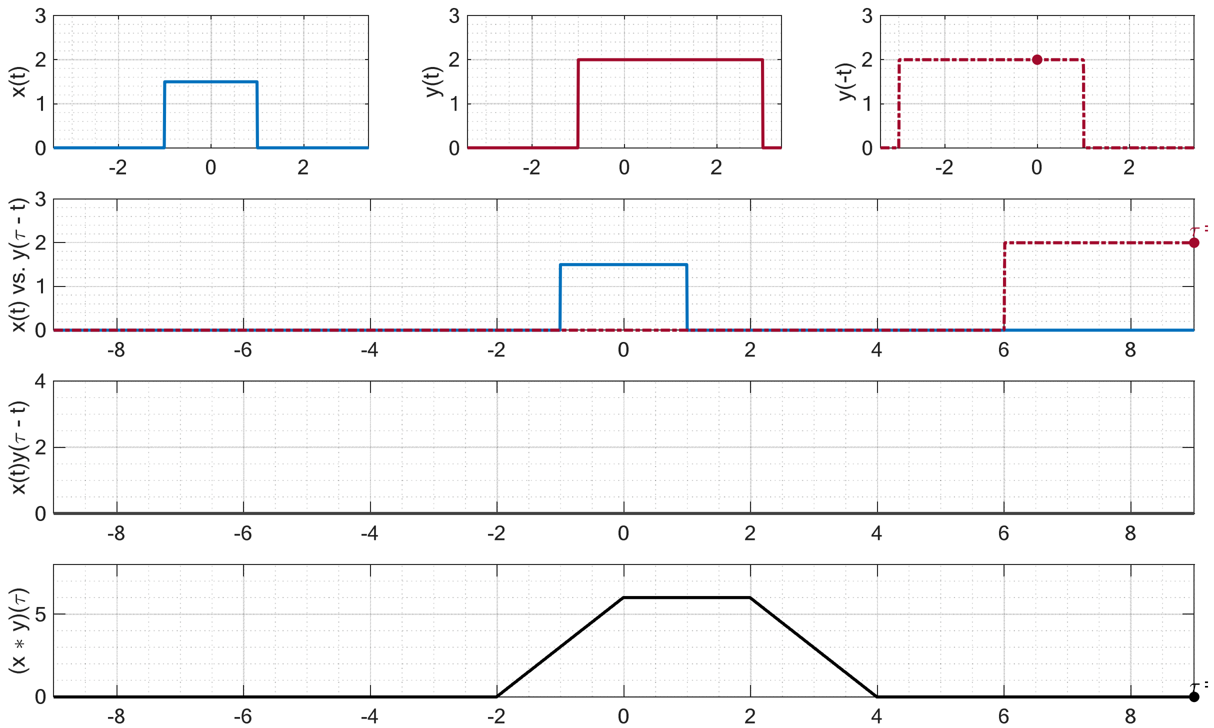
$$z(\tau) = x(t) * y(t) = \int_{-\infty}^{\infty} x(t)\, y(\tau - t)\, dt$$

$$z(\tau) = \int_{-\infty}^{\infty} (0)\, dt = 0$$

Full Solution (trapezoid):

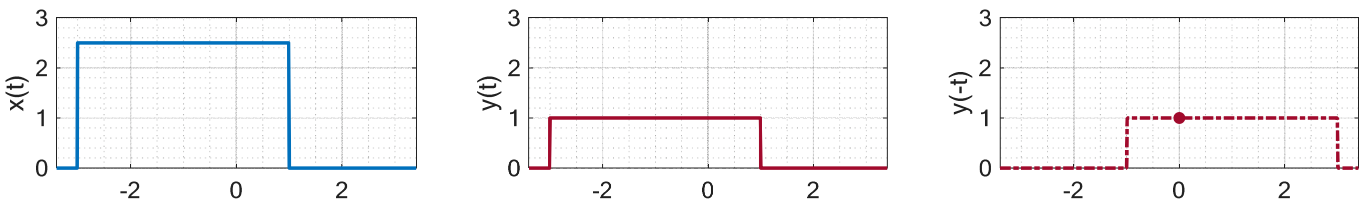
$$z(\tau) = x(t) * y(t) = \begin{cases} 0, & \tau \leq -2 \\ 3\tau + 6, & -2 < \tau \leq 0 \\ 6, & 0 < \tau \leq 2 \\ -3\tau + 12, & 2 < \tau \leq 4 \\ 0, & \tau > 4 \end{cases}$$

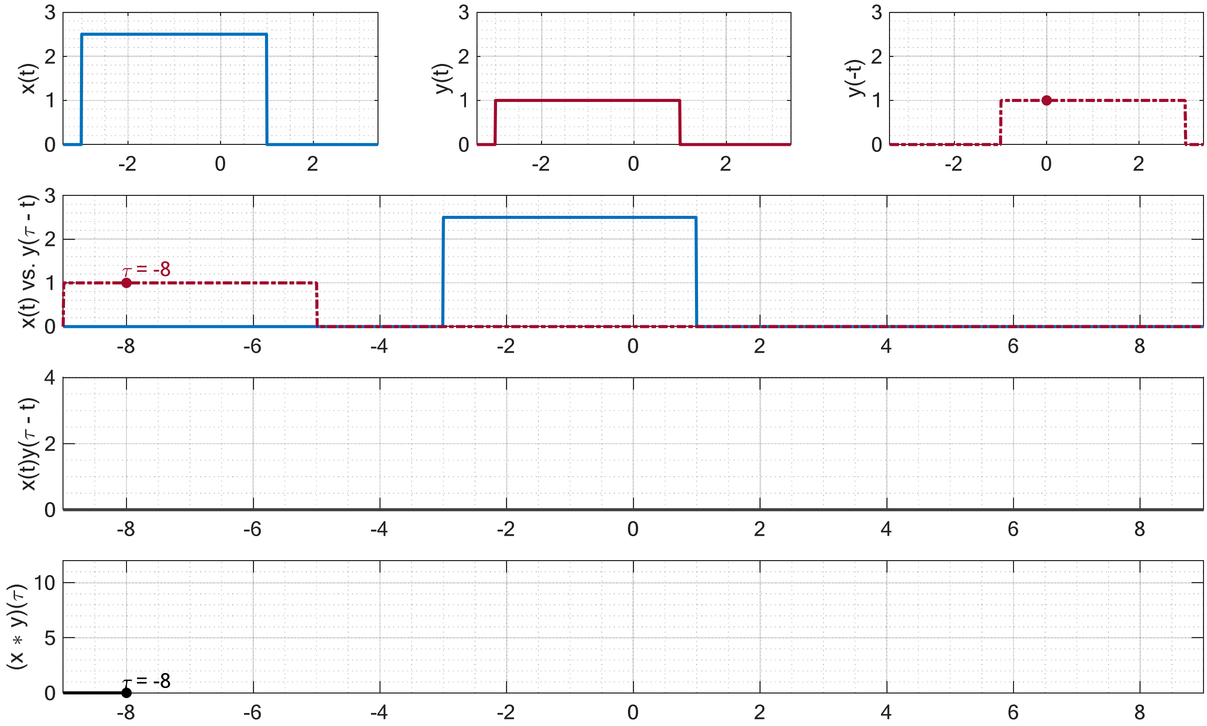


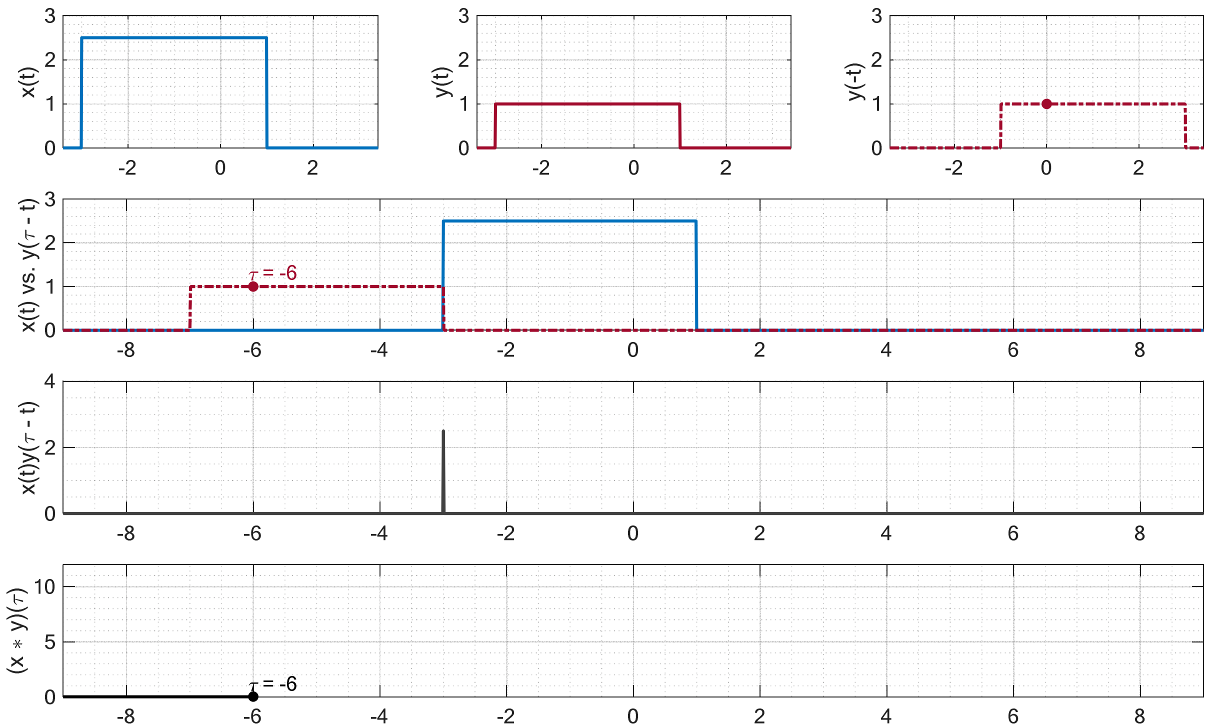


Q4. For the signals $x(t) = 2.5 \text{ rect}\left(\frac{t+1}{4}\right)$ and $y(t) = \text{rect}\left(\frac{t+1}{4}\right)$, determine the convolution result $x(t) * y(t)$.

Q4. Solution. To perform the graphical solution, first draw $x(t)$, $y(t)$ and $y(-t)$ as follows




x(t)
y(t)
y(-t)
x(t) vs. y(τ - t)
τ = -8
x(t)y(τ - t)
(x ∗ y)(τ)
τ = -8


x(t)
y(t)
y(-t)
x(t) vs. y(τ - t)
τ = -6
x(t)y(τ - t)
(x ∗ y)(τ)
τ = -6

Region #1: For time-shift $\tau \leq -6$

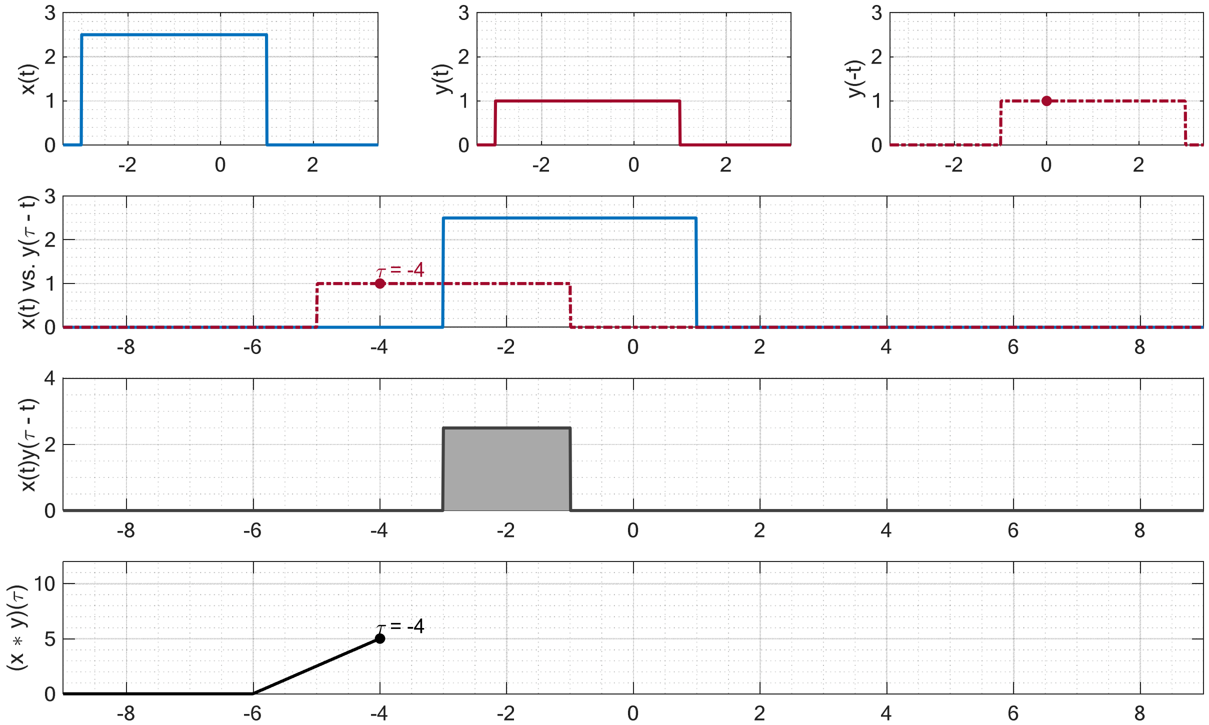
$$z(\tau) = x(t) * y(t) = \int_{-\infty}^{\infty} x(t)\, y(\tau - t)\, dt$$

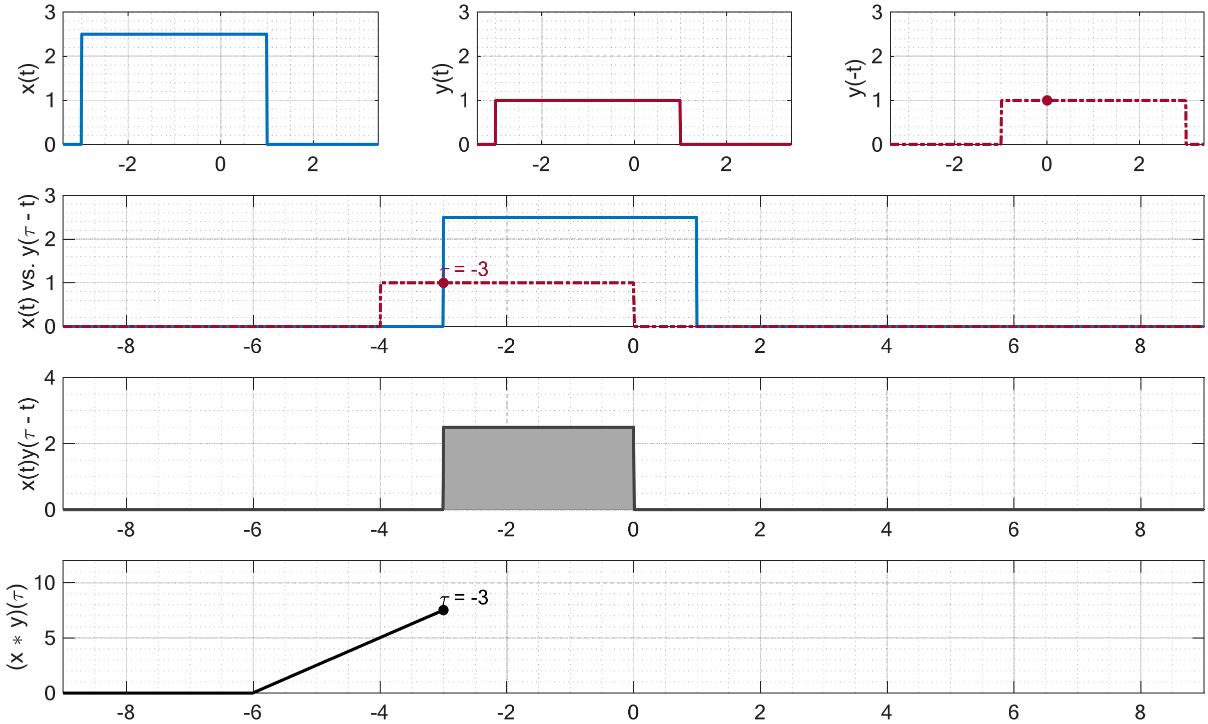
$$z(\tau) = \int_{-\infty}^{\infty} (0)\, dt = 0$$

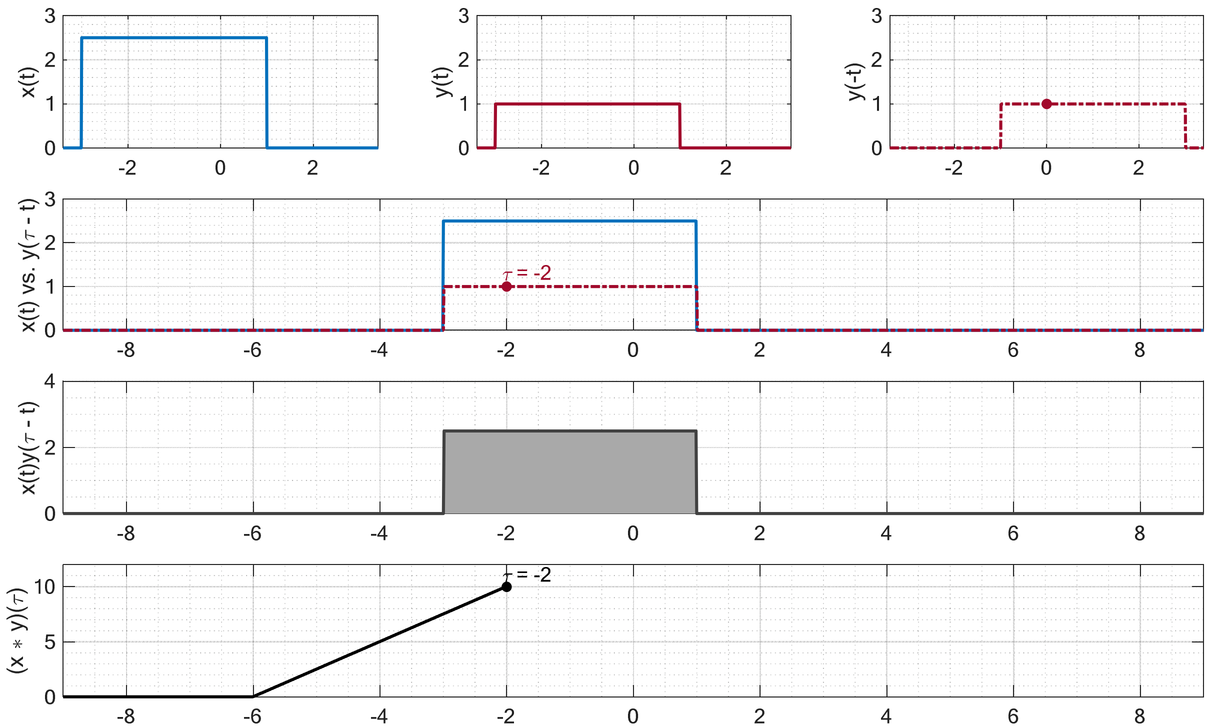
Region #2: For time-shift $-6 < \tau \leq -2$

$$z(\tau) = x(t) * y(t) = \int_{-\infty}^{\infty} x(t)\, y(\tau - t)\, dt$$

$$z(\tau) = \int_{-3}^{\tau+3} 2.5 \times 1\, dt = 2.5 \times 1 \times [t]_{-3}^{\tau+3} = 2.5(\tau + 6) = 2.5\tau + 15$$




x(t)
y(t)
y(-t)
x(t) vs. y(τ - t)
τ = -3
x(t)y(τ - t)
(x * y)(τ)
τ = -3


x(t)
y(t)
y(-t)
x(t) vs. y(τ - t)
τ = -2
x(t)y(τ - t)
(x * y)(τ)
τ = -2

Region #3: For time-shift $-2 < \tau \leq 2$

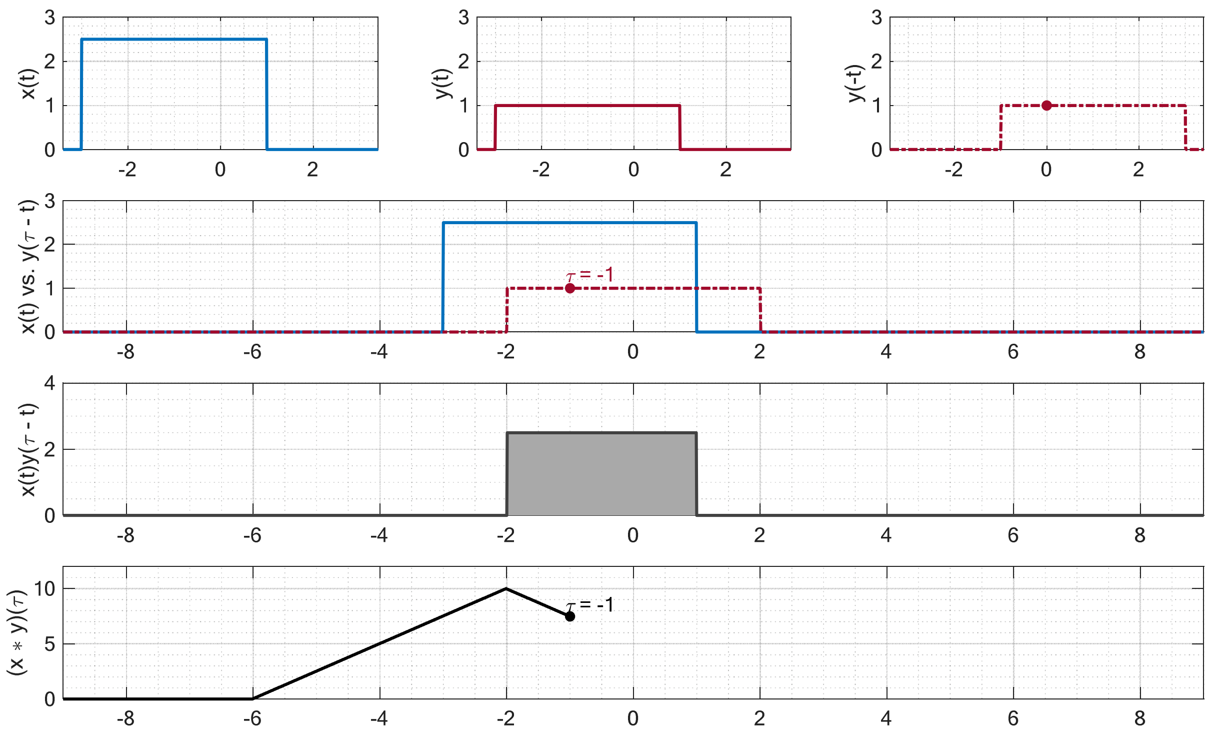
$$z(\tau) = x(t) * y(t) = \int_{-\infty}^{\infty} x(t)\, y(\tau - t)\, dt$$

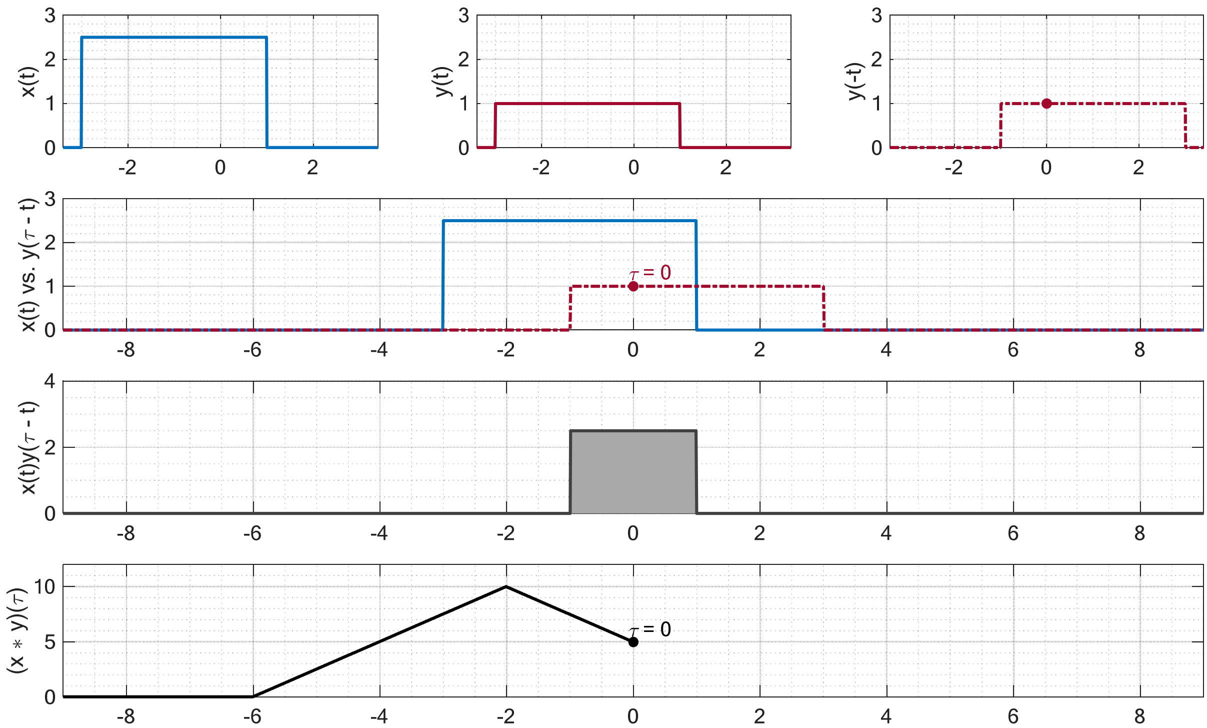
$$z(\tau) = \int_{\tau-1}^{1} 2.5 \times 1\, dt = 2.5 \times 1 \times [t]_{\tau-1}^{1} = 2.5(-\tau + 2) = -2.5\tau + 5$$

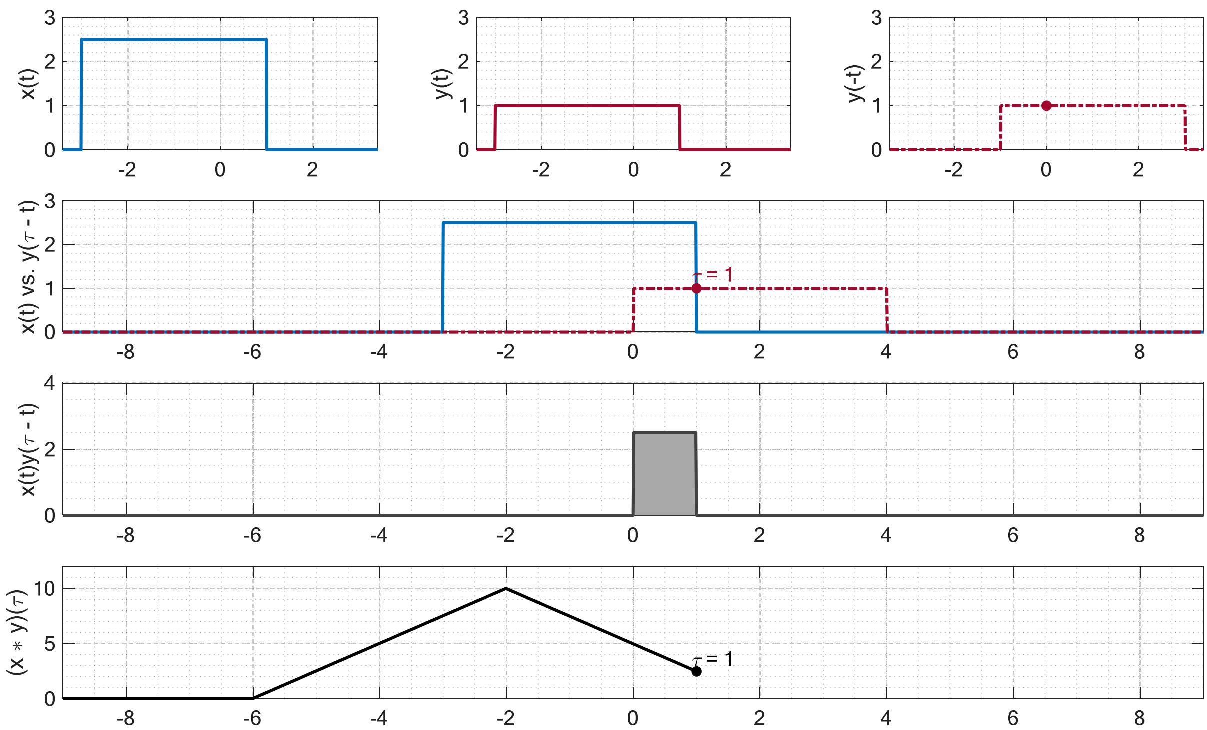
Region #4: For time-shift $\tau > 2$

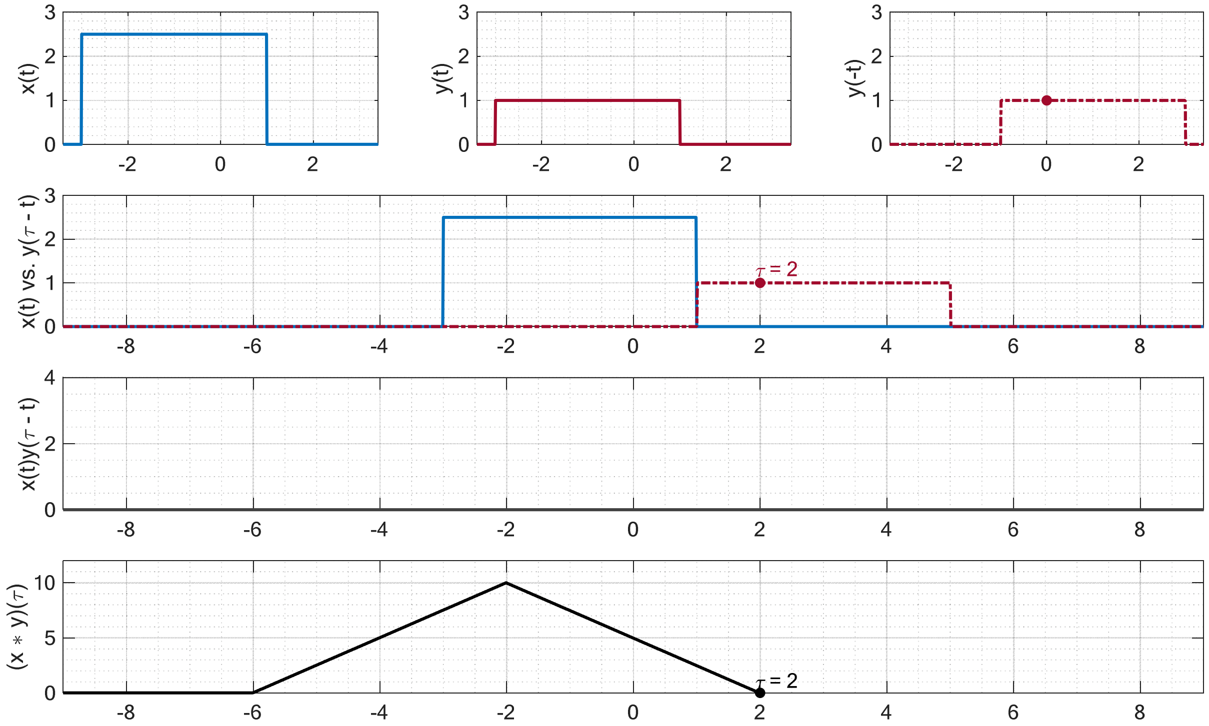
$$z(\tau) = x(t) * y(t) = \int_{-\infty}^{\infty} x(t)\, y(\tau - t)\, dt$$

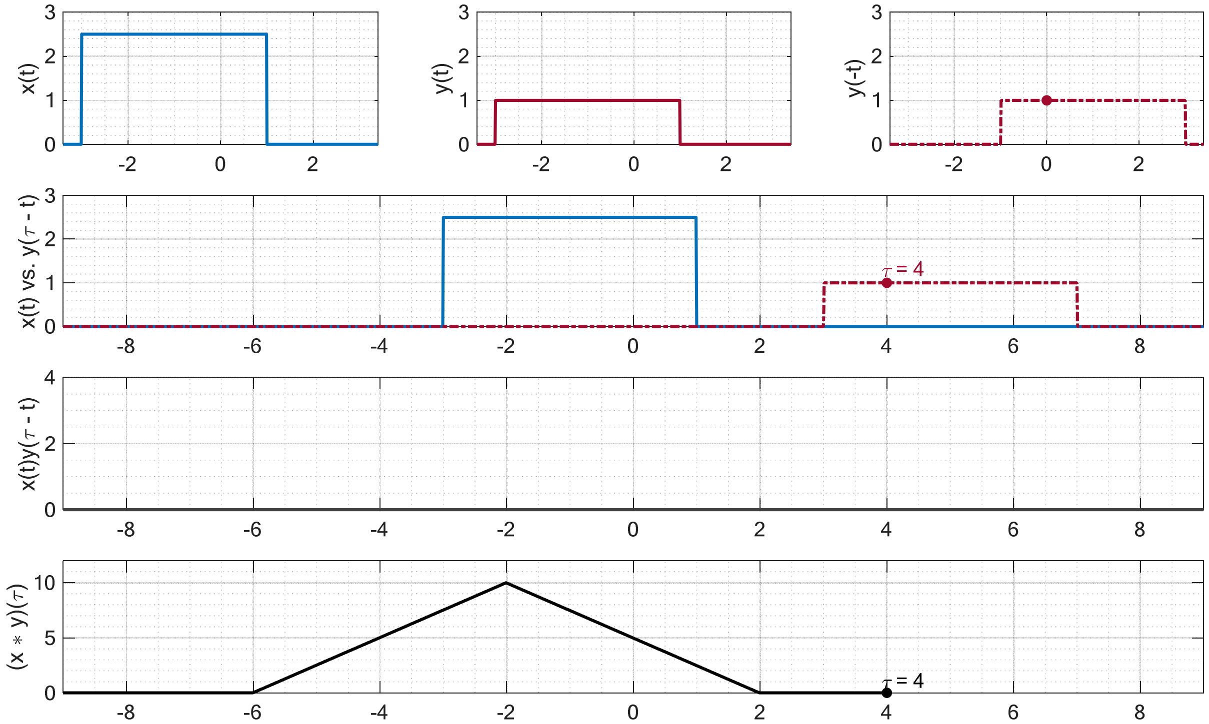
$$z(\tau) = \int_{-\infty}^{\infty} (0)\, dt = 0$$

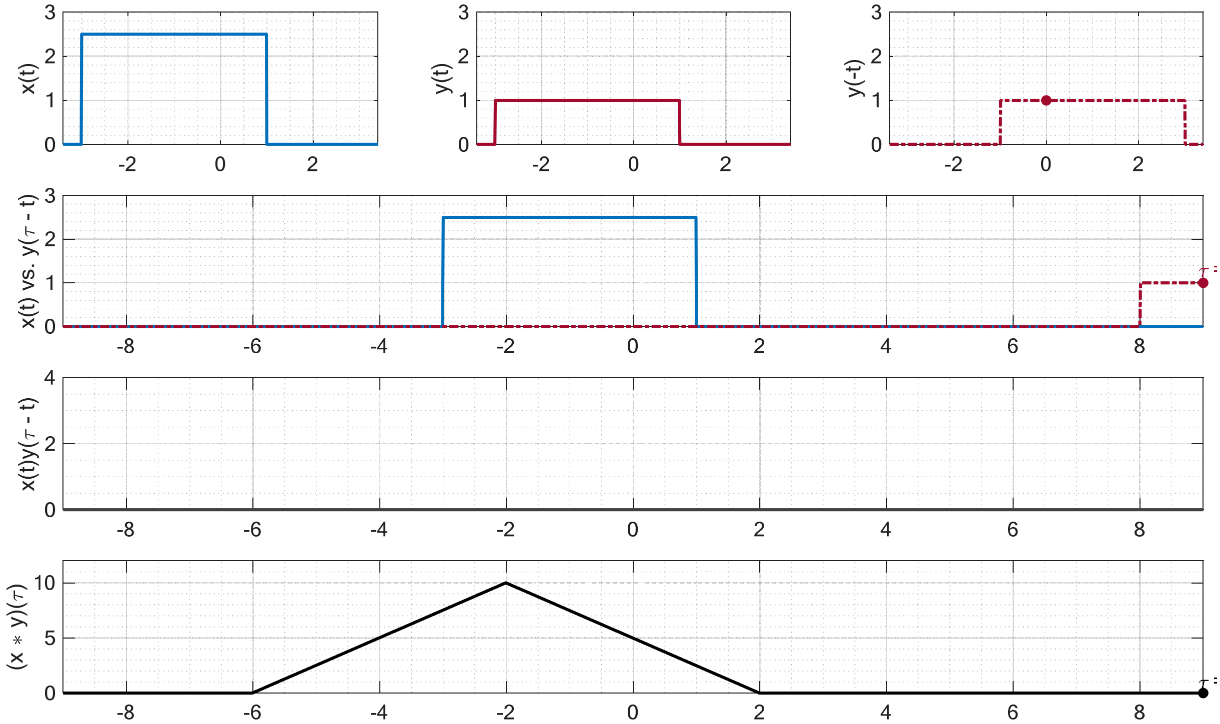



x(t)
y(t)
y(-t)
x(t) vs. y(τ - t)
τ = 0
x(t)y(τ - t)
(x * y)(τ)
τ = 0


x(t)
y(t)
y(-t)
x(t) vs. y(τ - t)
τ = 1
x(t)y(τ - t)
(x * y)(τ)
τ = 1


x(t)
y(t)
y(-t)
x(t) vs. y(τ - t)
τ = 2
x(t)y(τ - t)
(x ∗ y)(τ)
τ = 2


x(t)
y(t)
y(-t)
x(t) vs. y(τ - t)
τ = 4
x(t)y(τ - t)
(x ∗ y)(τ)
τ = 4



Full Solution (triangle):

$$z(\tau) = x(t) * y(t) = \begin{cases} 0, & \tau \leq -6 \\ 2.5\tau + 15, & -6 < \tau \leq -2 \\ -2.5\tau + 5, & -2 < \tau \leq 2 \\ 0, & \tau > 2 \end{cases}$$